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The Padé Tables of Entire Functions*

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INTRODUCTION

Let

$$f(z) = \sum_{m=0}^{\infty} a_m z^m \qquad (a_0 \neq 0) \tag{1}$$

have a radius of convergence σ_0 $(0 < \sigma_0 \leqslant +\infty)$.

The entries of the Padé table of (1) are ratios of polynomials which may be represented explicitly in terms of the Hankel determinants introduced below.

Let (m, n) be a pair of nonnegative integers; put

$$a_{-j} = 0 \qquad (j = 1, 2, 3,...) \tag{2}$$

and consider the determinants

$$A_m^{(n)} = \begin{vmatrix} a_m & a_{m-1} & \cdots & a_{m-n+1} \\ a_{m+1} & a_m & \cdots & a_{m-n+2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m+n-1} & a_{m+n-2} & \cdots & a_m \end{vmatrix}, \qquad A_m^{(0)} = 1, \tag{3}$$

$$W_m^{(n+1)}(j) = \begin{vmatrix} a_j & a_{j-1} & a_{j-2} & \cdots & a_{j-n} \\ a_{m+1} & a_m & a_{m-1} & \cdots & a_{m-n+1} \\ a_{m+2} & a_{m+1} & a_m & \cdots & a_{m-n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m+n} & a_{m+n-1} & a_{m+n-2} & \cdots & a_m \end{vmatrix}, \tag{4}$$

and the polynomials

$$D_{mn}(z) = \begin{vmatrix} 1 & z & z^2 & \cdots & z^n \\ a_{m+1} & a_m & a_{m-1} & \cdots & a_{m-n+1} \\ a_{m+2} & a_{m+1} & a_m & \cdots & a_{m-n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m+n} & a_{m+n-1} & a_{m+n-2} & \cdots & a_m \end{vmatrix}, \qquad D_{m0}(z) \equiv 1. \tag{5}$$

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With the above notations, the fundamental identity

$$f(z)\, D_{mn}(z) - \sum_{j=0}^{m} W_m^{(n+1)}(j)\, z^j = \sum_{j=m+n+1}^{\infty} W_m^{(n+1)}(j)\, z^j, \tag{6}$$

constitutes a natural starting point for the study of Padé approximation. It is obvious that the radius of convergence of the series in (6) is at least equal to the radius of convergence of the original series (1). It is often useful to express the right-hand side of (6) as a contour integral [5; pp. 436–437] and to use the resources of function theory to obtain its analytic continuation.

This aspect of the question need not concern us here since we always assume that $f(z)$ is entire.

Some additional notations will simplify our exposition. Put

$$W_{mn}(z) = \sum_{j=0}^{m} W_m^{(n+1)}(j)\, z^j. \tag{7}$$

Whenever $A_m^{(n)} \neq 0$, we also introduce the normalized Padé polynomials

$$\begin{aligned} P_{mn}(z) &= \frac{W_{mn}(z)}{A_m^{(n)}} = \sum_{j=0}^{m} p_j(m, n)\, z^j \qquad (P_{mn}(0) = a_0), \\ Q_{mn}(z) &= \frac{D_{mn}(z)}{A_m^{(n)}} = \sum_{j=0}^{n} q_j(m, n)\, z^j \qquad (Q_{mn}(0) = 1). \end{aligned} \tag{8}$$

For our purpose the coefficients of the leading terms of $P_{mn}(z)$ and $Q_{mn}(z)$ are important:

$$p_m(m, n) = \frac{A_m^{(n+1)}}{A_m^{(n)}}, \qquad q_n(m, n) = \frac{A_{m+1}^{(n)}}{A_m^{(n)}}. \tag{9}$$

The Padé polynomials P_{mn}, Q_{mn} are obvious generalizations of the notion of partial sum (or section) of the power series (1). One may expect that the distribution of the zeros of the Padé polynomials will be described by theorems analogous to the classical results of Jentzsch, Szegö, Carlson and Rosenbloom.

I have shown elsewhere [6] that, if $\sigma_0 < +\infty$, that is if the series (1) has a finite radius of convergence, the analogy with the theorems of Jentzsch and Szegö is complete.

If $f(z)$ is entire ($\sigma_0 = +\infty$) the corresponding questions cannot be treated as easily.

The main difficulty is to find (for suitable values of m and n) good lower bounds for $| A_m^{(n)} |$ as well as good bounds (from above and from below) for $| A_m^{(n+1)}/A_m^{(n)} |$.

In order to obtain such bounds, I consider, beside (1), all the power series

$$f_n(z) = \sum_{m=0}^{\infty} A_m^{(n)} z^m \qquad (n = 1, 2, 3, \ldots) \tag{10}$$

and apply to each one of them the Wiman–Valiron theory of the maximum term.

The key lemma of my investigation may be stated as follows.

LEMMA 1. *Let $f(z)$ be entire and transcendental.*
Denote by $\mu_n(r)$ the maximum term of the series in (10), *that is*

$$\max_j(|A_j^{(n)}| r^j) = \mu_n(r), \tag{11}$$

and write $\mu(r)$ instead of $\mu_1(r)$.

I. *Then, for all $r > 0$, we have*

$$\mu_n(r^n) \leqslant n^n(\mu(r))^n \qquad (n = 1, 2, 3, \ldots). \tag{12}$$

II. *With every integer $n \geqslant 1$, it is possible to associate an exceptional set E (of finite logarithmic measure) such that*

$$\mu_n(r^n) \geqslant (\mu(r))^n(\log \mu(r))^{-n(n-1)/2}(\log \log \mu(r))^{-5n(n-1)/2} \quad (r \notin E). \tag{13}$$

Remark 1. As an immediate consequence of (12) and (13), we obtain

$$\log \mu_n(r^n) \sim n \log \mu(r) \qquad (r \to \infty, r \notin E). \tag{14}$$

It is easily deduced from the above relation that, if $f(z)$ is of order exactly λ $(0 \leqslant \lambda \leqslant +\infty)$, then $f_n(z)$ is of order exactly λ/n.

Remark 2. In the special case $f(z) = e^z$, Stirling's formula and the explicit values of the Hankel determinants [1; p. 12, formula (3.7)]:

$$A_m^{(n)}[e^z] = A_m^{(n)} = \prod_{k=1}^{n} \frac{(k-1)!}{(m+k-1)!} \qquad (m \geqslant n \geqslant 1),$$

lead to the relations

$$K_n'\{\log \mu(r)\}^{n(n-1)/2} \leqslant \frac{\{\mu(r)\}^n}{\mu_n(r^n)} \leqslant K_n''\{\log \mu(r)\}^{n(n-1)/2} \qquad (r > r_0(n)),$$

where K_n' and K_n'' are positive constants depending only on n.

This shows that the inequality (13) is not far from best possible.

Lemma 1 leads to my main

THEOREM 1. *Let* (1) *represent an entire function of order* λ $(0 < \lambda \leqslant +\infty$. *Then, with every* $n \geqslant 1$, *it is possible to associate an unbounded sequence* $S(n)$ *of strictly increasing, positive integers which behaves as follows.*

I. *For* $m \in S(n)$ *we have*

$$A_m^{(n)} A_m^{(n+1)} \neq 0. \tag{15}$$

II. *Let* φ_1 *and* φ_2 *be given* $(\varphi_1 < \varphi_2 < \varphi_1 + 2\pi)$ *and let* $N(m; \varphi_1, \varphi_2)$ *denote the number of zeros of* $P_{mn}(z)$ *in the angle*

$$\varphi_1 \leqslant \arg z \leqslant \varphi_2 .$$

Then, if $\lambda < +\infty$, *we have*

$$N(m; \varphi_1, \varphi_2) \geqslant m\Omega \qquad (m \in S(n)), \tag{16}$$

where we may choose

$$\Omega = \frac{1}{5}\left\{\frac{\sin(\varphi_2 - \varphi_1)/12}{11 + (3/\lambda)}\right\}^{\omega}, \qquad \omega = \exp\left(\frac{\pi\{9 + (3/\lambda)\}}{\varphi_2 - \varphi_1}\right). \tag{17}$$

III. *If* $\lambda = +\infty$, *the inequality* (16) *may be replaced by the equidistribution relation*

$$\frac{N(m; \varphi_1, \varphi_2)}{m} \to \frac{\varphi_2 - \varphi_1}{2\pi} \qquad (m \to \infty,\ m \in S(n)). \tag{18}$$

IV. *For* $0 < \lambda \leqslant +\infty$ *it is possible to associate, with each* $m \in S(n)$, *a radius* R_m *such that, as* $m \to \infty$, $m \in S(n)$, *we have*

$$\begin{aligned} &R_m \to +\infty && (\lambda = +\infty), \\ &R_m > m^{1/(\lambda+\epsilon)} && (\lambda < +\infty,\ \epsilon > 0,\ \epsilon \text{ arbitrary}), \end{aligned} \tag{19}$$

as well as

$$R_m \left| \frac{A_m^{(n+1)}}{A_m^{(n)}} \right|^{1/m} \to e^{1/\Lambda} \qquad (e^{1/\Lambda} = 1,\ \text{if } \lambda = +\infty). \tag{20}$$

The quantity Λ, *which may depend on the choice of* $S(n)$, *satisfies the condition*

$$\lambda \leqslant \Lambda \qquad (e^{1/\Lambda} = 1 \text{ for } \lambda = +\infty). \tag{21}$$

If $\lambda = +\infty$ *and* $\eta > 0$ *is given, there are, as* $m \to \infty$ $(m \in S(n))$, *no more than* $o(m)$ *zeros of* $P_{mn}(z)$ *in each one of the regions*

$$|z| \geqslant R_m(1 + \eta), \tag{22}$$

and

$$|z| \leqslant R_m(1+\eta)^{-1}. \tag{23}$$

If $\lambda < +\infty$, (22) *is unaffected but it is necessary to replace* (23) *by*

$$|z| \leqslant R'_m, \tag{24}$$

where $\{R'_m\}$ *is any positive sequence such that* $(R'_m/R_m) \to 0$ $(m \to \infty, m \in S(n))$.

V. *If* $\lambda < +\infty$, *there are no more than*

$$\frac{m}{1+\lambda\log(1+\eta)}(1+o(1)) \qquad (m \to \infty,\ m \in S(n)), \tag{25}$$

zeros of $P_{mn}(z)$ *in the disk*

$$|z| \leqslant R_m e^{-1/\Lambda}(1+\eta)^{-1} \qquad (\eta > 0).$$

The remarkably precise form of assertion III is obtained by applying, to a suitable sequence of polynomials, a classical result of Erdös and Turán [9]. The method yields more than has been asserted above and leads to the following modified form of assertion III:

IIIa. *If* $\varphi_2 - \varphi_1 > 0$ *is given, and if*

$$\lambda > 2(32)^2 \frac{\pi^2}{(\varphi_2-\varphi_1)^2}, \tag{26}$$

then

$$N(m;\varphi_1,\varphi_2) > \frac{m}{2}\left(\frac{\varphi_2-\varphi_1}{2\pi}\right) \qquad (m \in S(n),\ m > m_0). \tag{27}$$

If λ is small (in the sense that (26) is violated) the theorem of Erdös and Turán may no longer be applicable. In this case, we use instead, in exactly the same way, the less precise Theorem 3 (stated at the end of this Introduction).

Theorem 1 above does not guarantee the existence of large disks in which some sequence of approximants of the nth row of the Padé table converges to the entire function $f(z)$. Suitably adapted, the arguments which lead to Theorem 1 also yield

THEOREM 2. *Let* $f(z)$ *be entire, transcendental, of finite order* $\lambda \geqslant 0$, *and let* $n \geqslant 1$ *be an integer such that*

$$0 < 1 - \lambda\frac{n(n-1)}{2} = \xi \leqslant 1. \tag{28}$$

It is then possible to find an infinite sequence $\mathscr{S}(n)$ of positive, strictly increasing integers having the following properties.

I. *If* $\lambda > 0$ *and* $m \in \mathscr{S}(n)$, *then*

$$| Q_{mn}(z) - 1 | \leqslant m^{-\xi/3\lambda}, \tag{29}$$

throughout the disk

$$| z | \leqslant m^{\xi/3\lambda}. \tag{30}$$

In the smaller disk

$$| z | \leqslant e^{-1} m^{\xi/3\lambda}, \tag{31}$$

we have

$$| f(z) Q_{mn}(z) - P_{mn}(z) | \leqslant \exp(-m + m^{\xi/2}). \tag{32}$$

II. *If* $\lambda = 0$, *the quantity* $m^{\xi/3\lambda}$, *which appears in* (29), (30) *and* (31), *may be remlaced by* m^B, *with* $B > 0$ (*B is otherwise arbitrary*). *The form of* (32) *is not affected.*

The convergence properties of the rows of the Padé immediately follow from Theorem 2. We thus find

COROLLARY 2.1. *If* $\lambda = 0$, *we may, with each* $n \geqslant 1$, *associate a sequence* $\mathscr{S}(n)$ *such that*

$$\left| f(z) - \frac{P_{mn}(z)}{Q_{mn}(z)} \right| \leqslant \exp(-m + m^{1/2}), \tag{33}$$

provided

$$| z | \leqslant m^B, \qquad m \in S(n). \tag{34}$$

It is clearly possible to state a similar Corollary for the positive values of λ which satisfy (28).

The first two rows of the Padé table of an entire function were successfully investigated by Baker [2]. Theorems 6 and 8 of Baker's paper are essentially equivalent to the assertions concerning $n = 1$ and $n = 2$ in Theorem 2 of the present paper.

Baker [2; p. 523] conjectured the possibility of studying the convergence of all the rows of the Padé table of an entire function. Theorem 2 above shows that, with simple limitations on the order of $f(z)$, it is possible to carry out Baker's program.

Theorems 1 and 2 require no regularity assumptions concerning the growth of $f(z)$ or the coefficients of its expansion.

It is to be expected that suitable regularity restrictions will make it possible to eliminate the sequences $S(n)$ and $\mathscr{S}(n)$ from Theorems 1 and 2 and to assert, instead, that inequalities such as (16), (29) and (32) hold for all sufficiently large values of m.

A striking illustration of the implications of regularity may be derived from the study of special choices of $f(z)$. Among the noteworthy examples of this kind, I mention the elegant and very precise results of Saff and Varga [13] concerning the zeros of the Padé polynomials associated with the exponential function.

My proof of assertion II of Theorem 1 makes essential use of the following

THEOREM 3. *Let*

$$T(z) = 1 + t_1 z + t_2 z^2 + \cdots + t_m z^m = \prod_{j=1}^{m} \left(1 - \frac{z}{\zeta_j}\right) \tag{35}$$

be a polynomial such that

$$|t_m| = e^{-\alpha m} \qquad (\alpha \geqslant 0, m \geqslant 1), \tag{36}$$

and such that

$$\max_{|z|=1} |T(z)| \leqslant e^{\eta m} \qquad (0 < \eta). \tag{37}$$

Denote by $\mathcal{N}$ the number of zeros of $T(z)$ in the angle

$$\varphi_1 \leqslant \arg z \leqslant \varphi_2 \qquad \left(0 < \varphi_2 - \varphi_1 = \frac{\pi}{\gamma}, \gamma > \frac{1}{2}\right).$$

Define

$$\kappa = \kappa(\alpha, \gamma) = \frac{1}{4} \left\{\frac{\sin(\pi/12\gamma)}{11 + 3\alpha}\right\}^{\omega} \qquad (\omega = \exp(\gamma(2\alpha + 9))). \tag{38}$$

Then, if

$$\eta \leqslant \kappa, \tag{39}$$

we have

$$\mathcal{N} \geqslant m\kappa. \tag{40}$$

THEOREM OF ERDÖS AND TURÁN. *With the normalization and notations of Theorem* 3:

$$\left|\frac{\mathcal{N}}{m} - \frac{1}{2\gamma}\right| < 16 \left\{\frac{1}{m} \log(1 + |t_1| + |t_2| + \cdots + |t_m|) + \frac{\alpha}{2}\right\}^{1/2}. \tag{41}$$

In the above inequality the logarithm is necessarily positive, hence (41) will only give a non-trivial lower bound for $\mathcal{N}/m$ provided

$$\alpha < 2(32\gamma)^{-2}.$$

The validity of Theorem 3 is not restricted by a condition of this type. On the other hand, a comparison of (40) and (41) shows that the value of κ (given by (38)) is strikingly unfavorable.

Results having the same general character as the above Theorem 3 were discovered and announced by Carlson [3], in 1924. He did not publish his proofs before 1948 [4]. The first proofs to appear in the literature were given by Rosenbloom [11] in his remarkable thesis (1943). A summary of some of his results and proofs will be found in [12]. The theorem of Erdös and Turán [9] was published in 1950.

For the convenience of the reader I sketch a brief proof of Theorem 3 based on Rosenbloom's ideas. The use of an identity of Edrei and Fuchs [8; pp. 144–145] leads without difficulty to the simple explicit bound expressed by (38) and (40).

1. Terminology and notations

Sets E, of finite logarithmic measure, play an important role in this paper. By definition E, which is a set of values of the positive quantity r, satisfies the two following conditions:

(i) it is a measurable subset of the interval

$$1 \leqslant r < +\infty;$$

(ii) $\int_E (dr/r) < +\infty$.

By K_n, I denote a positive constant which only depends on the integer $n \geqslant 1$ and on no other parameters (neither explicitly nor implicitly).

By $\{\eta_m\}_m$, I denote a sequence (not necessarily positive) such that

$$\eta_m \to 0 \qquad (m \to \infty).$$

Symbols such as $(r > r_0)$, $(m > m_0)$,... following some relation, mean that the relation in question holds for sufficiently large values of the parameters $r, m, \ldots$.

The symbol E does not represent the same set at each occurrence. Similarly K_n, $\{\eta_m\}$, r_0, m_0,..., may have different values in different places.

It is essential for the understanding of the following proofs to bear in mind that our conventions imply that the union of a finite number of sets E is still E and that $r \notin E$ and $r^n \notin E$ are equivalent assertions.

2. Power Series Whose Coefficients are Hankel Determinants

In everything that follows (1) defines an entire transcendental function. We denote by $\mu(r)$ and $\nu(r)$, respectively, the maximum term and the central index of the expansion (1):

$$\mu(r) = \max_{0 \leq j} \{|a_j| r^j\},$$

and $\nu(r)$ is the largest integer $j \geq 0$ such that

$$\mu(r) = |a_j| r^j.$$

Since

$$A_j^{(1)} = a_j$$

we have, as in Lemma 1 stated in the Introduction,

$$\mu(r) = \mu_1(r).$$

The elementary identity [10; p. 5, ex. 33]

$$\log \mu(r) - \log \mu(1) = \int_1^r \frac{\nu(t)}{t} \, dt, \tag{2.1}$$

is fundamental and will be taken for granted.

Lemma 2.1. *Let $f(z)$ be entire and transcendental. The determinants $W_m^{(n+1)}(j)$, $D_{mn}(z)$ and the polynomial $W_{mn}(z)$ defined by* (4), (5) *and* (7) *satisfy the following relations.*

I. *For any $r > 0$ and any nonnegative integers m, n, j, we have*

$$|W_m^{(n+1)}(j)| \leq (n+1)^{n+1} (\mu(r))^{n+1} r^{-mn-j}, \tag{2.2}$$

which imply

$$|A_m^{(n+1)}| \leq (n+1)^{n+1} \left(\frac{\mu(r)}{r^m}\right)^{n+1} \tag{2.3}$$

and

$$\max_{|z| \leq r} |W_{mn}(z)| \leq (m+1)(n+1)^{n+1} (\mu(r))^{n+1} r^{-mn}. \tag{2.4}$$

II. *If in addition, we assume $0 \leq t < r$, then*

$$\max_{|z| \leq t} |f(z) D_{mn}(z) - W_{mn}(z)| \leq (n+1)^{n+1} \frac{(\mu(r))^{n+1}}{r^{mn}} \left(\frac{t}{r}\right)^{m+n} \frac{t}{r-t}. \tag{2.5}$$

We also have

$$\max_{|z|\leqslant t} | D_{mn}(z) - A_m^{(n)} | \leqslant n^n \left(\frac{\mu(r)}{r^m}\right)^n \frac{t}{r} \left(1 + \frac{t}{r}\right)^{n-1}, \tag{2.6}$$

for $t \geqslant 0$, *and without the restriction* $t < r$.

III. *Each one of the functions* $f_n(z)$ *in* (10) *is entire and transcendental.*

Proof. Number the rows and columns of the determinant $W_m^{(n+1)}(j)$ from 0 to n. Multiply the lth row by r^{m-n+l} and perform this operation for $l = 1, 2, 3,..., n$. [The 0th row remains unaltered.]

Similarly multiply by r^{n-k} the kth column of $W_m^{(n+1)}(j)$ ($k = 0, 1, 2,..., n-1$).

These operations transform the lth row of $W_m^{(n+1)}(j)$ into

$$a_{m+l}r^{m+l}, a_{m+l-1}r^{m+l-1},..., a_{m+l-n}r^{m+l-n} \qquad (l \geqslant 1) \tag{2.7}$$

and the 0th row into

$$a_j r^n, a_{j-1}r^{n-1},..., a_{j-n+1}r, a_{j-n} . \tag{2.8}$$

The sum of the moduli of all the terms in (2.7) cannot exceed

$$(n+1)\,\mu(r).$$

The analogous sum formed with the members of (2.8) cannot exceed

$$r^{n-j}(n+1)\,\mu(r).$$

It is now obvious that

$$r^{(m+1)n} \,|\, W_m^{(n+1)}(j)| \leqslant (n+1)^{n+1} (\mu(r))^{n+1} r^{n-j} \tag{2.9}$$

and (2.2) follows.

The above inequality could be slightly improved by the use of Hadamard's estimate of the modulus of a determinant; such an improvement would not be reflected in our Theorems 1 and 2.

The inequalities (2.3) and (2.4) immediately follow from (2.2) and from our definitions (3) and (7). Taking (6) into account we also obtain (2.5).

To prove (2.6), we consider

$$\begin{vmatrix} 0 & z & z^2 & \cdots & z^n \\ a_{m+1} & a_m & & \cdots & a_{m-n+1} \\ a_{m+2} & a_m & & \cdots & a_{m-n+2} \\ \cdots & \cdots & & \cdots & \cdots \\ a_{m+n} & a_{m+n-1} & & \cdots & a_m \end{vmatrix} = D_{mn}(z) - A_m^{(n)}$$

instead of $W_m^{(n+1)}(j)$. The arguments which lead to (2.9) now yield

$$r^{(m+1)n} \mid D_{mn}(z) - A_m^{(n)} \mid \leqslant n^n(\mu(r))^n \sum_{k=1}^{n} \mid z \mid^k r^{n-k}$$

and (2.6) follows.

To prove assertion III of the Lemma, notice that (2.3), with r and n fixed, implies

$$\limsup_{m\to\infty} \mid A_m^{(n+1)} \mid^{1/m} \leqslant r^{-n-1} \qquad (n \geqslant 0). \tag{2.10}$$

Since $r > 0$ is arbitrary, the right-hand side of (2.10) may be replaced by 0. Hence $f_{n+1}(z)$ is entire. It cannot reduce to a polynomial. Otherwise, by a known property of Hankel determinants [10; p. 103, ex. 23], we would conclude, against our assumption, that $f(z)$ is a rational function.

The proof of Lemma 2.1 is now complete.

3. Application of the Results of Wiman-Valiron

We make use of the very simple form given by Saxer [14; pp. 210–219] to the results of Wiman–Valiron. Throughout the remainder of this paper $\mu_n(r)$ and $\nu_n(r)$ denote, respectively, the maximum term and the central index of the series defining $f_n(z)$ (in (10)).

In addition to the results of Wiman–Valiron our proofs require the following modified form of a growth lemma of E. Borel.

Lemma 3.1. *Let $G(r) \geqslant e$ be a nondecreasing, unbounded function defined for all $r \geqslant r_0 \geqslant 1$.*

Then, if r lies outside a suitable exceptional set E, of finite logarithmic measure, we have

$$G\left(r\left\{1 + \frac{1}{\{\log G(r)\}^2}\right\}\right) < eG(r) \qquad (r \notin E).$$

Proof. The preceding statement follows immediately from a known lemma of Edrei and Fuchs. Adopting the notations of [7], we apply [7; p. 339, Lemma 10.1] with

$$h(x) = x^{-2}, \qquad \phi(x) = \log G(e^x), \qquad c = 1$$

and express the resulting inequalities in terms of the variable $r = e^x$.

As pointed out by an anonymous referee, Lemma 3.1 follows, just as simply, from a result of R. Nevanlinna [Bull. Sci. Math. 55 (1931), 140–144].

We now prove Lemma 1 stated in the Introduction.

Proof of assertion I. From (2.3) we deduce

$$(r^n)^m \mid A_m^{(n)} \mid \leqslant n^n(\mu(r))^n, \tag{3.1}$$

where $r > 0$ may be chosen arbitrarily. With the particular choice

$$m = \nu_n(r^n), \tag{3.2}$$

(3.1) reduces to (12).

Proof of assertion II. In order to apply the lemma of Wiman–Valiron we set

$$A_j^{(n)} = \alpha_j \qquad (j = 0, 1, 2, 3, \ldots)$$

and let the integer $n \geqslant 1$ be fixed. The function

$$\Xi(t) = \sum_{j=0}^{\infty} \alpha_j t^j \qquad (\mid t \mid = r^n)$$

is, by assertion III of Lemma 2.1, an entire function of the auxiliary variable t.
Introduce the sequence

$$\pi_{m+1} = \prod_{j=2}^{m} \left(1 + \frac{1}{j(\log j)^2}\right) \qquad (m = 2, 3, 4, \ldots). \tag{3.3}$$

The sequence $\{\pi_m\}$ thus defined is to some extent arbitrary; for our purpose it is important to note that

$$\frac{\pi_{m+1}}{\pi_m} > 1 \quad (m = 3, 4, 5, \ldots), \qquad \lim_{m \to \infty} \pi_m < +\infty.$$

Our notations and (3.2) imply

$$m = \nu_n(\mid t \mid)$$

and the inequalities of Wiman–Valiron are valid if

$$\mid t \mid = r^n \notin E, \tag{3.4}$$

where E is a suitable set (of values of $\mid t \mid \geqslant 1$) of finite logarithmic measure.
By definition, our notations yield

$$\mid \alpha_m \mid \mid t \mid^m = \mu_n(\mid t \mid) = \mu_n(r^n) = \mid A_m^{(n)} \mid r^{mn}, \tag{3.5}$$

$$\mid \alpha_{m-1} \mid \mid t \mid^{m-1} = \mid A_{m-1}^{(n)} \mid r^{(m-1)n} \leqslant \mid \alpha_m \mid \mid t \mid^m = \mid A_m^{(n)} \mid r^{mn}, \tag{3.6}$$

and, by the lemma of Wiman–Valiron [14; pp. 213–216],

$$|\alpha_{m+1}|\,|t|^{m+1} < \frac{\pi_m}{\pi_{m+1}}\,|\alpha_m|\,|t|^m \qquad (|t| \notin E).$$

The latter inequality, which we put in the form

$$|A_{m+1}^{(n)}|\,r^{(m+1)n} < \frac{\pi_m}{\pi_{m+1}}\,|A_m^{(n)}|\,r^{mn}, \tag{3.7}$$

is, in different notation, the first of the inequalities II, p. 213, of Saxer's statement [14].

From (3.6) and (3.7) we deduce

$$|A_{m-1}^{(n)}A_{m+1}^{(n)}| \leqslant \frac{\pi_m}{\pi_{m+1}}\,|A_m^{(n)}|^2, \tag{3.8}$$

which we use in the fundamental identity [10; p. 102, ex. 19]

$$A_m^{(n+1)}A_m^{(n-1)} = \{A_m^{(n)}\}^2 - A_{m-1}^{(n)}A_{m+1}^{(n)}.$$

We thus find

$$|A_m^{(n+1)}A_m^{(n-1)}| \geqslant |A_m^{(n)}|^2\left(\frac{\pi_{m+1}-\pi_m}{\pi_{m+1}}\right) \qquad (n \geqslant 1,\ A_m^{(0)} = 1),$$

and, in view of (3.3),

$$|A_m^{(n+1)}A_m^{(n-1)}| \geqslant \frac{|A_m^{(n)}|^2}{2m(\log m)^2} \qquad (m \geqslant m_0). \tag{3.9}$$

Hence, using (3.5), we find

$$\mu_{n+1}(r^{n+1})\,\mu_{n-1}(r^{n-1}) \geqslant (\mu_n(r^n))^2\,2^{-1}m^{-1}(\log m)^{-2} \qquad (r \notin E). \tag{3.10}$$

Put

$$G(r) = \log \mu_n(r), \qquad R = r + \frac{r}{(\log G(r))^2}.$$

By (2.1) and Lemma 3.1

$$\nu_n(r)\,\frac{R-r}{R} < (e-1)\,G(r) \qquad (r \notin E),$$

and hence, by (12) (already proved),

$$m = \nu_n(r^n) < K_n \log \mu(r)(\log\log \mu(r))^2 \qquad (r \notin E). \tag{3.11}$$

From (3.10) and (3.11) we deduce

$$\log \mu_{n+1}(r^{n+1}) - 2 \log \mu_n(r^n) + \log \mu_{n-1}(r^{n-1}) > \kappa(r) \qquad (r \notin E), \tag{3.12}$$

where we may choose

$$\kappa(r) = -\log \log \mu(r) - 5 \log \log \log \mu(r).$$

The first member of (3.12) has the character of a second difference; moreover, with a proper definition of E, and the convention

$$\log \mu_0(r^0) = 0,$$

inequalities such as (3.12) are simultaneously valid for

$$n = 1, 2, \ldots, N - 1 \qquad (2 \leqslant N < +\infty,\ E = E(N)).$$

Performing two successive summations, we deduce from (3.12)

$$\log \mu_n(r^n) > n \log \mu(r) + \frac{n(n-1)}{2} \kappa(r) \qquad (r \notin E),$$

and (13) follows. The proof of Lemma 1 is now complete.

4. Bounds for the Maximum Modulus and for the Leading Term of $P_{mn}(z)$

Lemma 4.1. *Let the notations and assumptions of Lemma* 1 *be unchanged. Then, taking*

$$m = \nu_n(r^n), \qquad r \notin E, \tag{4.1}$$

we have $A_m^{(n)} \neq 0$, *and*

$$\max_{|z|=r} | P_{mn}(z)| \leqslant \{\mu_n(r^n)\}^{1/n} \{\log \mu_n(r^n)\}^{K_n} (m + 1), \tag{4.2}$$

$$r^m \left| \frac{A_m^{(n+1)}}{A_m^{(n)}} \right| \geqslant \{\mu_n(r^n)\}^{1/n} (\log \mu_n(r^n))^{-K_n} m^{-2}. \tag{4.3}$$

Proof. By (4.1), m is a central index of the expansion of $f_n(z)$; hence $A_m^{(n)} \neq 0$. By (8) and (2.4)

$$\max_{|z|=r} | P_{mn}(z)| = \max_{|z|=r} \left| \frac{W_{mn}(z)}{A_m^{(n)}} \right|$$

$$\leqslant (m + 1) K_n \frac{(\mu(r))^{n+1}}{| A_m^{(n)} | r^{mn}} = (m + 1) K_n \frac{(\mu(r))^{n+1}}{\mu_n(r^n)}. \tag{4.4}$$

Similarly, by (3.9),

$$r^m \left| \frac{A_m^{(n+1)}}{A_m^{(n)}} \right| r^{m(n-1)} \mid A_m^{(n-1)} \mid \geqslant m^{-2} \mid A_m^{(n)} \mid r^{mn} \qquad (r \notin E),$$

$$r^m \left| \frac{A_m^{(n+1)}}{A_m^{(n)}} \right| \mu_{n-1}(r^{n-1}) \geqslant m^{-2} \mu_n(r^n) \qquad (r \notin E),$$

and taking (12), into account, we deduce

$$r^m \left| \frac{A_m^{(n+1)}}{A_m^{(n)}} \right| \geqslant m^{-2} K_n \frac{\mu_n(r^n)}{(\mu(r))^n} \mu(r) \qquad (r \notin E). \tag{4.5}$$

Using (12), (13) and (14) in (4.4) and (4.5), we obtain (4.2) and (4.3). This completes the proof of the lemma.

5. Proof of Theorem 2

An inspection of (28) shows that it is always possible to determine y such that

$$1 - y\lambda \frac{n(n-1)}{2} > \frac{9}{10} \xi, \qquad y > 1. \tag{5.1}$$

Having thus selected y we deduce from Lemma 1 the existence of some exceptional set E that, for $r \notin E$, (13) holds in the form

$$\frac{(\mu(r))^n}{\mu_n(r\ )} \leqslant (\log \mu(r))^{yn(n-1)/2} \qquad (r \notin E), \tag{5.2}$$

and (14) in the form

$$\log \mu_n((2r)^n) \sim n \log \mu(2r) \qquad (r \to \infty, r \notin E). \tag{5.3}$$

By the definition of order it is possible to find a function $\eta(r)$ such that

$$\eta(r) > 0, \qquad \eta(r) \to 0 \qquad (r \to \infty), \tag{5.4}$$

and

$$\log \mu(r) \leqslant \max_{|z|=r} \log |f(z)| \leqslant r^{\lambda+\eta(r)}. \tag{5.5}$$

We now choose

$$m = \nu_n(r^n) = m(r) \tag{5.6}$$

and, from the identity (2.1) deduce

$$\nu_n(r^n)\, n \log 2 \leqslant \int_{r^n}^{(2r)^n} \frac{\nu_n(t)}{t}\, dt \leqslant \log \mu_n((2r)^n) \qquad (r > r_0). \tag{5.7}$$

Using (5.3), (5.5) and (5.6) in (5.7) we find

$$m(r) \leqslant \frac{2}{\log 2} \log \mu(2r) \leqslant 2^{\lambda+2} r^{\lambda+\eta(r)} \qquad (r \notin E,\ r > r_0). \tag{5.8}$$

Select

$$t = t(r) = 2^{-4(\lambda+2)\xi/9(\lambda+\eta(r))} m^{4\xi/9(\lambda+\eta(r))}, \tag{5.9}$$

so that, by (5.8),

$$t \leqslant r^{4\xi/9} \leqslant r \qquad (r > 1). \tag{5.10}$$

From (2.6)

$$| Q_{mn}(z) - 1 | \leqslant \frac{t}{r}\left(1 + \frac{t}{r}\right)^{n-1} n^n \frac{(\mu(r))^n}{\mu_n(r^n)} \qquad (| z | \leqslant t),$$

and hence, choosing t as in (5.9), and taking (5.2) and (5.10) into account, we find

$$| Q_{mn}(z) - 1 | \leqslant \frac{t}{r} K_n(\log \mu(r))^{\gamma n(n-1)/2} \qquad (r \notin E,\ | z | \leqslant t). \tag{5.11}$$

We now use (5.5), (5.1) and (5.10) in (5.11); this yields

$$| Q_{mn}(z) - 1 | \leqslant r^{-4\xi/9} \qquad (r \notin E,\ r > r_0,\ | z | \leqslant t). \tag{5.12}$$

To eliminate r from (5.12) we use (5.8) and obtain

$$| Q_{mn}(z) - 1 | \leqslant 2^{4(\lambda+2)\xi/9(\lambda+\eta(r))} m^{-4\xi/9(\lambda+\eta(r))} \leqslant m^{-7\xi/18(\lambda+\eta(r))}$$
$$(| z | \leqslant t,\ m = m(r),\ r \notin E,\ m > m_0). \tag{5.13}$$

The relations (5.13) also hold (by (5.9)) for

$$| z | \leqslant m^{7\xi/18(\lambda+\eta(r))} \leqslant t \qquad (r > r_0). \tag{5.14}$$

Let $\{r_k\}_{k=1}^{\infty}$ be any positive sequence tending to infinity by values such that $r_k \notin E$.

The corresponding values $m(r_k)$, defined by (5.6), determine our sequence $\mathscr{S}(n)$.

If $\lambda > 0$, if $m \in \mathscr{S}(n)$, and if $m > m_0$, the inequality (30) implies (5.14). Hence (5.13) holds and (29) follows.

To prove (32), consider the fundamental relation between $P_{mn}(z)$ and $Q_{mn}(z)$, expressed as a contour integral [5; pp. 434–437]:

$$f(z)\, Q_{mn}(z) - P_{mn}(z) = \frac{z^{m+n+1}}{2\pi i} \int_{\mathscr{C}} \frac{f(\zeta)\, Q_{mn}(\zeta)}{\zeta^{m+n+1}(\zeta - z)}\, d\zeta. \tag{5.15}$$

Take $\mathscr{C}$ to be the circumference

$$|\, \zeta \,| = m^{\xi/3\lambda} \tag{5.16}$$

and

$$|\, z \,| \leqslant e^{-1}\, |\, \zeta \,|. \tag{5.17}$$

Notice also that, by (29),

$$\max_{\theta} |\, Q_{mn}(|\, \zeta \,|\, e^{i\theta})| \leqslant 2 \qquad (m > m_0), \tag{5.18}$$

and by (5.5) and (5.16)

$$\max_{\theta} |\, f(|\, \zeta \,|\, e^{i\theta})| \leqslant \exp(m^{\xi(\lambda+\eta(r))/3\lambda}) \leqslant \exp(m^{\xi/2}) \qquad (m > m_0). \tag{5.19}$$

Using (5.16)–(5.19) in (5.15), we obtain, by the familiar estimates for contour integrals,

$$|\, f(z)\, Q_{mn}(z) - P_{mn}(z)| \leqslant \exp(-m - n - 1 + m^{\xi/2})\, \frac{2}{1 - e^{-1}}$$

and (32) follows since $n \geqslant 1$.

To treat the case $\lambda = 0$ it suffices to set $\lambda = 0$ in (5.5), (5.8), (5.9), (5.13) and (5.14). We now have $\xi = 1$ and we may choose $y = 2$; the relation (5.16) is to be replaced by

$$|\, \zeta \,| = m^{7\xi/18\eta(r)}$$

and this value of $|\, \zeta \,|$ is to be used in (5.17).

The inequality (5.19) now takes the simple form

$$\max_{\theta} |\, f(|\, \zeta \,|\, e^{i\theta})| \leqslant \exp(m^{7\xi/18}).$$

The preceding considerations lead almost immediately to assertion II of Theorem 2 and to the Corollary 2.1.

6. The Ratio $\log \mu(r)/\nu(r)$

Lemma 6.1. *Let $f(z)$ be entire, of order λ $(0 < \lambda \leqslant +\infty)$ and let $\mu(r)$ and $\nu(r)$ be the maximum term and central index of the expansion* (1) *of $f(z)$.*

Then

$$\liminf_{r\to\infty} \frac{\log \mu(r)}{\nu(r)} \leqslant \frac{1}{\lambda}. \tag{6.1}$$

In the case $\lambda = +\infty$, *the right-hand side of* (6.1) *is to be interpreted as* 0.

Proof. For $\lambda < +\infty$ the inequality (6.1) is contained in ex. 60 of [10; p. 9] For $\lambda = +\infty$ we may argue as follows. The definition of the central index shows that $\nu(t)$ is constant on each interval

$$\tau_j \leqslant t < \tau_{j+1} \qquad (j = 1, 2, 3,...), \tag{6.2}$$

where $\{\tau_j\}_{j=1}^{\infty}$ is a strictly increasing, positive unbounded sequence. Hence, given $K > 0$, we have, in each of the intervals (6.2),

$$\frac{\nu(t)}{t^K} \leqslant \frac{\nu(\tau_j)}{\tau_j{}^K} = \rho_j . \tag{6.3}$$

The sequence $\{\rho_j\}_{j=1}^{\infty}$ thus defined is unbounded because otherwise there would exist an M $(0 < M < +\infty)$ such that

$$\nu(t) \leqslant Mt^K \qquad (t > t_0),$$

and, by (2.1), this would imply that the order of $f(z)$ does not exceed K. Hence it is clearly possible to find an infinite sequence

$$J = \{j_s\}_{s=1}^{\infty}$$

of positive, strictly increasing integers such that, if $j \in J$, we have

$$\rho_k \leqslant \rho_j \qquad (1 \leqslant k \leqslant j). \tag{6.4}$$

From (6.3) and (6.4) we conclude that

$$\nu(t) \leqslant t^K \frac{\nu(\tau_j)}{\tau_j{}^K} \qquad (\tau_1 \leqslant t \leqslant \tau_j). \tag{6.5}$$

Using (6.5) in (2.1) we find

$$\log \mu(\tau_j) \leqslant \log \mu(\tau_1) + \nu(\tau_j)\, \tau_j^{-K} \int_{\tau_1}^{\tau_j} t^{K-1}\, dt$$

$$\limsup_{\substack{j\to\infty \\ j\in J}} \frac{\log \mu(\tau_j)}{\nu(\tau_j)} \leqslant \frac{1}{K}. \tag{6.6}$$

Since $\lambda = +\infty$, the argument may be repeated for arbitrarily large values of K and we thus find

$$\liminf_{r\to\infty} \frac{\log \mu(r)}{\nu(r)} = 0.$$

It is obvious that with minor modifications the same reasoning leads to (6.1) in the case $\lambda < +\infty$.

The presence, in Lemma 4.1 of the exceptional set E makes it necessary, for the applications that we have in mind, to formulate

LEMMA 6.2. *Let the notations and assumptions of Lemma* 6.1 *be unchanged and let* $\delta > 0$ *be given.*

It is then possible to find a positive, strictly increasing, unbounded sequence $\{t_j\}_{j=1}^{\infty}$ *such that, in each one of the intervals*

$$t_j \leqslant t \leqslant t_j e^{\delta/2} \qquad (j = 1, 2, 3, \ldots), \tag{6.7}$$

we have

$$\frac{\log \mu(t)}{\nu(t)} \leqslant \frac{1}{\lambda} + \delta. \tag{6.8}$$

Proof. From Lemma 6.1 we deduce the existence of $\{t_j\}$ such that

$$\frac{\log \mu(t_j)}{\nu(t_j)} < \frac{1}{\lambda} + \frac{\delta}{2} \qquad (j = 1, 2, 3, \ldots). \tag{6.9}$$

Then, by (2.1)

$$\log \mu(t) = \log \mu(t_j) + \int_{t_j}^{t} \frac{\nu(x)}{x}\, dx \leqslant \log \mu(t_j) + \nu(t)\frac{\delta}{2}$$
$$(t_j \leqslant t \leqslant t_j e^{\delta/2}). \tag{6.10}$$

Since $\nu(t) \geqslant \nu(t_j)$, (6.8) follows from (6.9) and (6.10). The argument remains valid in the limiting case $\lambda = +\infty$.

7. SELECTION OF THE SEQUENCE $S(n)$ OF THEOREM 1

Let $n \geqslant 1$ be a given integer which remains fixed throughout the proof.

The only entire function to be considered is $f_n(z)$ whose order is exactly λ/n $(0 < \lambda \leqslant +\infty)$. [This fact is established in the Remark following the statement of Lemma 1.]

All our inequalities are expressed in terms of $\mu_n(r^n)$ and $\nu_n(r^n)$ and E is the exceptional set such that for $r \notin E$, the relations (3.11), (4.2) and (4.3) are all valid.

Assume that the first $k - 1$ members of $S(n)$

$$m_1, m_2, \ldots, m_{k-1} \qquad (k \geqslant 2) \tag{7.1}$$

have been chosen.

Apply Lemma 6.2 to $f_n(z)$, with $\delta = 1/k$. This establishes the existence of infinitely many intervals

$$I_j = (t_j^n, t_j^n e^{n/2k})$$

such that, if $t^n \in I_j$, then

$$\frac{\log \mu_n(t^n)}{\nu_n(t^n)} \leqslant \frac{n}{\lambda} + \frac{n}{k}. \tag{7.2}$$

There are intervals I_j with arbitrarily large initial points t_j^n and the logarithmic measure of each I_j is $n/2k$. Hence, it will be possible to find some I_j and some r_k such that:

$$m_{k-1} < \nu_n(t_j^n), \qquad r_k^n \in I_j, \qquad r_k \notin E,$$

and, by (7.2),

$$\frac{\log \mu_n(r_k^n)}{\nu_n(r_k^n)} \leqslant \frac{n}{\lambda} + \frac{n}{k}. \tag{7.3}$$

Selecting

$$m_k = \nu_n(r_k^n), \tag{7.4}$$

we define the element of $S(n)$ which follows the initial elements (7.1). An obvious induction enables us to complete the construction of the infinite sequence $S(n)$. We now define

$$R_m = r_k \qquad (m = m_k \in S(n)) \tag{7.5}$$

and eliminate k from all our formulae.

With the new notation, (7.4) becomes

$$m = \nu_n(R_m^n) \qquad (m \in S(n)), \tag{7.6}$$

and (7.3) takes the form

$$\frac{\log y_m}{m} \leqslant \frac{n}{\lambda} + \eta_m \qquad (m \in S(n)), \tag{7.7}$$

where

$$y_m = \mu_n(R_m^n). \tag{7.8}$$

From (4.3), (7.6) and (7.8) we deduce

$$R_m^{\ m} \left| \frac{A_m^{(n+1)}}{A_m^{(n)}} \right| \geqslant y_m^{1/n}(\log y_m)^{-K_n}\, m^{-2} \qquad (m \in S(n)), \tag{7.9}$$

and, from (4.2),

$$\max_{|z|=R_m} |P_{mn}(z)| \leqslant y_m^{1/n}(\log y_m)^{K_n}\,(m+1) \qquad (m \in S(n)). \tag{7.10}$$

Cauchy's estimate, (7.10) and (8) imply

$$R_m^{\ m} \left| \frac{A_m^{(n+1)}}{A_m^{(n)}} \right| \leqslant y_m^{1/n}(\log y_m)^{K_n}\,(m+1) \qquad (m \in S(n)). \tag{7.11}$$

Define

$$\frac{1}{\Lambda_m} = \frac{\log y_m}{mn} > 0, \tag{7.12}$$

and notice that (7.7), (7.9) and (7.11) yield

$$0 \leqslant \limsup_{\substack{m\to\infty \\ m\in S(n)}} \frac{1}{\Lambda_m} \leqslant \frac{1}{\lambda}, \tag{7.13}$$

and

$$R_m \left| \frac{A_m^{(n+1)}}{A_m^{(n)}} \right|^{1/m} = e^{1/\Lambda_m}(1+\eta_m) \qquad (m \in S(n)). \tag{7.14}$$

Considering if necessary a subsequence $\Sigma(n)$ of $S(n)$ we deduce from (7.13)

$$\lim_{\substack{m\to\infty \\ m\in\Sigma(n)}} \frac{1}{\Lambda_m} = \frac{1}{\Lambda} \qquad (\lambda \leqslant \Lambda \leqslant +\infty), \tag{7.15}$$

and from (7.14)

$$R_m \left| \frac{A_m^{(n+1)}}{A_m^{(n)}} \right|^{1/m} \to e^{1/\Lambda} \qquad (m \to \infty,\ m \in \Sigma(n)). \tag{7.16}$$

We may now discard the sequence $S(n)$ originally constructed and write $S(n)$ instead of $\Sigma(n)$. With this new notation (7.10), (7.12) and (7.15) yield

$$\max_{|z|=R_m} \log |P_{mn}(z)| \leqslant \frac{m}{\Lambda} + o(m) \qquad (m \to \infty,\ m \in S(n)). \tag{7.17}$$

The fact that

$$R_m \to +\infty \qquad (m \to \infty,\ m \in S(n)) \tag{7.18}$$

is an obvious consequence of its definition. For $\lambda < +\infty$, we deduce from (3.11) an explicit lower bound for R_m:

$$m = \nu_n(R_m{}^n) < \{\log \mu(R_m)\}^{(1+\epsilon/3(\lambda+1))} < R_m^{(\lambda+\epsilon/3)(1+\epsilon/3(\lambda+1))},$$

$$m < R_m^{\lambda+\epsilon} \qquad (m > m_0\,,\ 0 < \epsilon < 1). \tag{7.19}$$

8. Proof of Assertions I, II, III and IIa of Theorem 1

Assertion I is an immediate consequence of (7.9), (7.11) and of our choice of $S(n)$.

It is convenient to introduce beside the polynomials $P_{mn}(z)$, polynomials $T_m(\zeta)$ defined by the relations

$$\begin{aligned} T_m(\zeta) &= R_m^{-m}\,\frac{A_m^{(n)}}{A_m^{(n+1)}}\,\zeta^m P_{mn}\left(\frac{R_m}{\zeta}\right) \\ &= t_{mm}\zeta^m + t_{m,m-1}\zeta^{m-1} + \cdots + t_{m1}\zeta + t_{m0}\,, \end{aligned} \tag{8.1}$$

where

$$m \in S(n), \qquad t_{mm} = a_0 R_m^{-m}\,\frac{A_m^{(n)}}{A_m^{(n+1)}}\,, \qquad t_{m0} = 1. \tag{8.2}$$

By (7.16)

$$\lim_{\substack{m\to\infty \\ m\in S(n)}} |\,t_{mm}\,|^{1/m} = e^{-1/A}. \tag{8.3}$$

By (7.9), (7.10) and (7.12)

$$\max_{|\zeta|=1} |\,T_m(\zeta)| \leqslant 2m^3(\log y_m)^{K_n} \leqslant 2m^{3+K_n}\left(\frac{n}{A_m}\right)^{K_n}. \tag{8.4}$$

By Cauchy's estimate and our convention concerning K_n, we deduce from (8.4)

$$\sum_{j=0}^{m} |\,t_{mj}\,| < m^{K_n}A_m^{-K_n} \qquad (m \in S(n),\ m > m_0). \tag{8.5}$$

Put

$$\mathscr{E}_m = \frac{1}{m}\log\left(\sum_{j=0}^{m} |\,t_{mj}\,|\right) - \frac{1}{2m}\log|\,t_{mm}\,|; \tag{8.6}$$

by (8.3) and (8.5) we obtain

$$\mathscr{E}_m \leqslant \frac{1}{2A} + \eta_m + \frac{1}{m}(K_n \log m - K_n \log A_m) \leqslant \frac{1}{2A} + \eta_m\,. \tag{8.7}$$

Let φ_1, φ_2, $N(m; \varphi_1, \varphi_2)$ be the quantities in the statement of Theorem 1; it is clear that $N(m; \varphi_1, \varphi_2)$ is also the number of zeros of $T_m(\zeta)$ in the angle

$$-\varphi_2 \leqslant \arg \zeta \leqslant -\varphi_1 \qquad \left(\frac{\varphi_2 - \varphi_1}{2\pi} = \frac{1}{2\gamma}\right).$$

Proof of assertion IIIa. We apply to $T_m(\zeta)$ the Theorem of Erdös and Turán. Using (8.6), we see that (36) implies

$$\left| \frac{N(m; \varphi_1, \varphi_2)}{m} - \frac{1}{2\gamma} \right| < 16\mathscr{E}_m^{1/2}, \tag{8.8}$$

where by (8.7) (and in view of our notational convention concerning η_m)

$$\mathscr{E}_m^{1/2} \leqslant \frac{1}{(2\Lambda)^{1/2}} + \eta_m .$$

Hence

$$\liminf_{\substack{m\to\infty \\ m\in S(n)}} \frac{N(m; \varphi_1, \varphi_2)}{m} \geqslant \frac{1}{2\gamma} - \frac{8(2)^{1/2}}{\Lambda^{1/2}} . \tag{8.9}$$

The additional assumption (26) implies

$$\Lambda^{1/2} \geqslant \lambda^{1/2} > 32(2)^{1/2}\gamma.$$

Consequently

$$\frac{1}{2\gamma} - \frac{8(2)^{1/2}}{\Lambda^{1/2}} = \frac{1}{2\gamma}\left(1 - \frac{16(2)^{1/2}\gamma}{\Lambda^{1/2}}\right) > \frac{1}{4\gamma} \tag{8.10}$$

and (27) follows from (8.9) and (8.10).

Proof of assertion III. By assumption $\Lambda = +\infty$, so that (8.7) implies

$$\mathscr{E}_m \to 0 \qquad (m \to \infty,\ m \in S(n)).$$

Hence (8.8) yields (18).

Proof of assertion II. We apply Theorem 3 to $T_m(\zeta)$.
By (8.4)

$$\max_{|\zeta|=1} |T_m(\zeta)| \leqslant \exp(\eta_m m), \tag{8.11}$$

and by (8.3)

$$|t_{mm}| = \exp\left(-\left(\frac{1}{\Lambda} + \eta_m\right) m\right), \qquad \Lambda \geqslant \lambda > 0.$$

An inspection of Theorem 3 shows that we may set in (36)

$$\alpha = \alpha_m = \frac{1}{\Lambda} + \eta_m$$

and conclude

$$N(m;\varphi_1, \varphi_2) \geqslant m\kappa(\alpha_m, \gamma) \qquad (m \in S(n)). \tag{8.12}$$

It is now obvious that

$$\kappa(\alpha_m, \gamma) > \Omega \qquad (m > m_0) \tag{8.13}$$

with Ω defined by (17).

Hence (16) follows from (8.12) and (8.13).

This completes the proof of assertion II of Theorem 1.

9. The Moduli of the Zeros; Proof of Assertions IV and V of Theorem 1

The radii R_m were introduced in (7.5). The relations (7.18) and (7.19) yield (19) and the relations (7.15) and (7.16) imply (20) and (21).

Consider the factored forms

$$P_{mn}(z) = a_0 \prod_{j=1}^{m} \left(1 - \frac{z}{z_j}\right) \qquad (z_j = z_j(m)), \tag{9.1}$$

and

$$T_m(\zeta) = t_{mm} \prod_{j=1}^{m} \left(\zeta - \frac{R_m}{z_j}\right). \tag{9.2}$$

From (9.2), Jensen's theorem and (8.11) we deduce

$$\sum_{|R_m/z_j| \leqslant (1+\eta)^{-1}} \log \frac{1}{|R_m/z_j|} \leqslant \eta_m m.$$

Hence, as $m \to \infty$ ($m \in S(n)$), there cannot be more than $o(m)$ zeros of $P_{mn}(z)$ in the regions (22).

Again, by Jensen's theorem, (9.1) and (7.17)

$$\log |a_0| + \sum_{|z_j| \leqslant R_m e^{-1/\Lambda}(1+\eta)^{-1}} \log \left| \frac{R_m}{z_j} \right| \leqslant \frac{m}{\Lambda} + o(m) \qquad (m \to \infty,\ m \in S(n))$$

and assertion V of Theorem 1 immediately follows.

With insignificant modifications the same arguments enable us to verify the assertions concerning (23) and (24).

The proof of Theorem 1 is now complete provided Theorem 3 is taken for granted.

10. Estimates for Functions Regular in a Sector

LEMMA 10.1. *Assume that $g(z)$ is regular in the angle*

$$\mathfrak{A}: \quad |\arg z| \leqslant \frac{\pi}{2c} \qquad \left(c > \frac{1}{2}\right)$$

and let

$$|g(z)| \leqslant M(R) \qquad (z \in \mathfrak{A}, |z| \leqslant R). \tag{10.1}$$

For u fixed, let

$$|g(ue^{i\theta})| \leqslant \beta \qquad \left(u > 0, \beta > 0, |\theta| \leqslant \frac{\pi}{2c}\right). \tag{10.2}$$

Assume also that, for some $\sigma > 0$,

$$1 \leqslant |g(ue^{\sigma+i\theta})| \qquad \left(\theta \leqslant \frac{\pi}{2c}\right). \tag{10.3}$$

Then

$$1 \leqslant \beta M_1^B, \tag{10.4}$$

where

$$M_1 = M(ue^{2\sigma}), \qquad B = e^{\sigma c} + e^{-\sigma c} - 1.$$

Proof. By [8; p. 145, formula (16)]

$$0 \leqslant \frac{2}{\pi} \log(\beta M_1) + \frac{2}{\pi} (e^{\sigma c} + e^{-\sigma c} - 2) \log M_1$$

and (10.4) follows.

11. Proof of Theorem 3

Let n be the number of zeros of $T(z)$ in the disk

$$\Gamma = \{z: |z| \leqslant \tfrac{1}{2}\}$$

and let N be the number of its zeros in the sector

$$\Delta = \{z: |z| > \tfrac{1}{2}, \varphi_1 \leqslant \arg z \leqslant \varphi_2\},$$

where

$$0 < \varphi_2 - \varphi_1 = \frac{\pi}{\gamma} \qquad \left(\gamma > \frac{1}{2}\right).$$

The normalization

$$\varphi_2 = -\varphi_1 = \frac{\pi}{2\gamma}$$

is always possible and, from this point on we assume that it has been performed.

By Jensen's formula, (37) and (39),

$$n \log 2 \leqslant \sum_{|\zeta_j| \leqslant 1/2} \log \left| \frac{1}{\zeta_j} \right| \leqslant \eta m \leqslant \kappa m. \tag{11.1}$$

Hence, since $\kappa < 1/44$,

$$n < \frac{m}{30}. \tag{11.2}$$

We now introduce three auxiliary polynomials

$$U(z) = \prod_{|\zeta_j| \leqslant 1/2} \left(1 - \frac{z}{\zeta_j}\right), \tag{11.3}$$

$$V(z) = \prod_{\zeta_j \in \Delta} \left(1 - \frac{z}{\zeta_j}\right), \tag{11.4}$$

$$X(z) = \frac{T(z)}{U(z)\, V(z)} = \prod_{j=1}^{h} \left(1 - \frac{z}{\xi_j}\right) = 1 + x_1 z + x_2 z + \cdots + x_h z^h, \tag{11.5}$$

and examine the behavior of $X(z)$ *under the additional assumption*

$$N < m\kappa. \tag{11.6}$$

By (11.2), (11.5) and (11.6) the exact degree of $X(z)$ is

$$h = m - n - N > (0.9)m. \tag{11.7}$$

From (11.3) we deduce

$$\min_{|z|=1} |U(z)| \geqslant 1, \tag{11.8}$$

and from (11.4)

$$\min_{|z|=1/4} |V(z)| \geqslant 2^{-N}. \tag{11.9}$$

Combining (11.5), (11.8) and (11.9), we find, in view of (37),

$$\max_{|z|=1/4} |X(z)| \leqslant 2^N \max_{|z|=1/4} \left|\frac{T(z)}{U(z)}\right| \leqslant 2^N \max_{|z|=1} \left|\frac{T(z)}{U(z)}\right| \leqslant 2^N e^{\eta m}. \qquad (11.10)$$

For the coefficient of the leading term of $X(z)$ we have

$$|x_h| = |t_m| \prod_{|\zeta_j| \leqslant 1/2} |\zeta_j| \prod_{\zeta_k \in \Delta} |\zeta_k|,$$

and hence by (36), (11.1), (11.6) and (11.7)

$$\begin{aligned} |x_h| &\geqslant \exp(-\alpha m - \kappa m - \kappa m \log 2) \\ &> \exp(-(10/9)\, h(\alpha + (1.7)\kappa)). \end{aligned} \qquad (11.11)$$

The polynomial $X(z)$ has no zeros in the set

$$\mathscr{L} = \{\Gamma \cup \Delta\};$$

consequently, the function $g(z)$ defined by the conditions

$$g(z) = \frac{1}{h} \log X(z), \qquad g(0) = 0, \qquad (11.12)$$

is holomorphic for $z \in \mathscr{L}$.

Bounds for $|g(z)|$ in the angle

$$|\arg z| \leqslant \frac{\pi}{3\gamma}$$

are immediate:

$$g(z) = \frac{1}{h} \sum_{j=1}^{h} \log\left(1 - \frac{z}{\xi_j}\right) = \frac{1}{h} \sum_{j=1}^{h} \int_0^z \frac{dw}{w - \xi_j},$$

$$|g(z)| \leqslant \frac{\delta}{h} \sum_{j=1}^{h} \int_0^{|z|} \frac{dx}{x + |\xi_j|} \qquad \left(\delta = \frac{1}{\sin(\pi/12\gamma)}\right),$$

$$|g(z)| \leqslant \delta \log(1 + 2|z|) \leqslant \delta(\log 2 + \log^+ 2|z|) \qquad \left(|\arg z| \leqslant \frac{\pi}{3\gamma}\right). \qquad (11.13)$$

By the Borel–Carathéodory inequality [15; pp. 174–175] and (11.10),

$$\max_{|z| \leqslant 1/8} |\log X(z)| \leqslant 2 \max_{|z| \leqslant 1/4} \log |X(z)| \leqslant 2(N \log 2 + m\eta),$$

and hence by (39), (11.6) and (11.7)

$$\max_{|z|\leqslant 1/8} |g(z)| \leqslant \frac{2}{h}(1 + \log 2)\,\kappa m < 4\kappa. \tag{11.14}$$

The Boutroux–Cartan lemma asserts that

$$|X(z)| \geqslant |x_h| \left(\frac{H}{4e}\right)^h,$$

outside exceptional disks with sum of diameters H.

Take

$$\log H = \log 4 + 2 + (10/9)\alpha + 2\kappa, \qquad u = \tfrac{1}{8},$$

so that, by (11.11),

$$\frac{1}{h}\log |X(ue^{\sigma+i\theta})| \geqslant 1 \qquad (|\theta| \leqslant \pi), \tag{11.15}$$

for some suitable value of σ such that

$$u < ue^{\sigma} < 2u + 4e^{2+(10/9)\alpha+2\kappa},$$

and therefore

$$0 < \sigma < 2 + (10/9)\alpha + 2\kappa + \log 34 < 5.6 + (10/9)\alpha = \sigma_1 . \tag{11.16}$$

Hence (11.13) yields

$$|g(ue^{2\sigma+i\theta})| \leqslant \delta(-\log 2 + 2\sigma_1) < \delta(11 + 3\alpha) \qquad \left(|\theta| \leqslant \frac{\pi}{3\gamma}\right). \tag{11.17}$$

By (11.12) and (11.15)

$$|g(ue^{\sigma+i\theta})| \geqslant 1. \tag{11.18}$$

In view of (11.18), Lemma 10.1 is immediately applicable to $g(z)$ with $c = 3\gamma/2$,

$$M_1 = \delta(11 + 3\alpha) > 1, \qquad \beta = 4\kappa, \tag{11.19}$$

$$\begin{aligned} B &= \exp(c\sigma) + \exp(-c\sigma) - 1 \\ &< \exp(c\sigma) < \exp\left(\frac{3\gamma}{2}\left(6 + \frac{10}{9}\alpha\right)\right) < \exp(\gamma(9 + 2\alpha)) = \omega. \end{aligned} \tag{11.20}$$

By (10.4), (11.19) and (11.20)

$$1 \leqslant \beta M_1{}^B < 4\kappa\{\delta(11 + 3\alpha)\}^{\omega} = 1. \tag{11.21}$$

[The last equality in (11.21) coincides with (38)]. We have thus obtained a contradiction which shows the impossibility of (11.6). Hence

$$\mathcal{N} \geqslant N \geqslant m\kappa,$$

and Theorem 3 is proved.

References

1. R. J. Arms and A. Edrei, The Padé tables and continued fractions generated by totally positive sequences, *in* "Mathematical Essays dedicated to A. J. Macintyre," pp. 1–21, Ohio Univ. Press, Athens, 1970.
2. G. A. Baker, Jr., The existence and convergence of subsequences of Padé approximants, *J. Math. Anal. Appl.* **43** (1973), 498–528.
3. F. Carlson, Sur les fonctions entières, *C. R. Acad. Sci. Paris* **179** (1924), 1583–1585.
4. F. Carlson, Sur les fonctions entières, *Ark. Mat. Astron. Fys.* **35A** (1948), 1–18.
5. A. Edrei, The Padé table of functions having a finite number of essential singularities, *Pacific J. Math.* **56** (1975), 429–453.
6. A. Edrei, Angular distribution of the zeros of Padé polynomials, *J. Approximation Theory* **24** (1978), 251–265.
7. A. Edrei and W. H. J. Fuchs, Bounds for the number of deficient values of certain classes of meromorphic functions, *Proc. London Math. Soc.* **12** (1962), 315–344.
8. A. Edrei and W. H. J. Fuchs, Asymptotic behavior of meromorphic functions with extremal spread, II, *Ann. Acad. Sci. Fenn.* **3** (1977), 141–168.
9. P. Erdös and P. Turán, On the distribution of roots of polynomials, *Ann. of Math.* **51** (1950), 105–119.
10. G. Pólya and G. Szegö, "Aufgaben und Lehrsätze aus der Analysis," Vol. 2, Springer-Verlag, Berlin, 1925.
11. P. C. Rosenbloom, "Sequences of polynomials, especially sections of power series," Ph.D. dissertation, Stanford, 1943.
12. P. C. Rosenbloom, Distribution of zeros of polynomials, *in* "Lectures on Functions of a Complex Variable" (W. Kaplan, Ed.), Univ. of Michigan Press, Ann Arbor, 1955.
13. E. B. Saff and R. S. Varga, On the zeros and poles of Padé approximants to e^z, *Numer. Math.*, **25** (1975), 1–14.
14. W. Saxer, Über die Picardschen Ausnahmewerte sukzessiver Derivierten, *Math. Z.* **17** (1923), 206–227.
15. E. C. Titchmarsh, "The Theory of Functions," 2nd ed., Oxford Univ. Press, London/New York, 1952.